Time Machines and the Breakdown of Unitarity

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Abstract

We present a generic way of thinking about time machines from the view of a far away observer. In this model the universe consists of three (or more) regions: One containing the entrance of the time machine, another the exit and the remaining one(s) the rest of the universe. In the latter we know ordinary quantum mechanics to be valid and thus are able to write down a Hamiltonian describing this generic time machine. We prove the time-evolution operator to be non-symmetric. Various interpretations of this irreversibility are given.

Introduction

The question of whether time machines are possible or not has been studied by several authors in the last couple of years. This interest was spawned by the realization that topologically non-trivial space-times may exhibit closed time-like curves, or "time machines". The most important example is an otherwise flat spacetime with a sufficiently short wormhole connecting two distant regions, see figure 1. This can be made to function as a time-machine, either by putting the two mouths of the wormhole in regions of different gravitational potential or by accelerating one with respect to the other, and then bringing it to rest again – both of these methods generate a time shift, which an object travelling through the wormhole experiences (Morris *et al.* (1988), Kim and Thorne (1991), Friedman *et al.* (1990), Novikov (1992)).
The presence of closed time-like curves (time-machines) would make the past and the future fuse in the sense that 'someone' travelling on a closed time-like curve could influence his own past (the past and future light cones overlap). So time-machines makes distinguishing past and future impossible, right? Wrong! The Hamiltonian describing the action of the time-machine becomes non-symmetric making the evolution operator non-unitary, and thus time machines will be time-asymmetric in a quantum mechanical context.
We can model such time machines very easily. First assume a 3+1 splitting of spacetime, i.e. the existence of a cosmic time (if the time machine is constructed from a wormhole, then this splitting will only be possible sufficiently far away from the mouths). Space will be divided into a number of regions. The time machine has its entrance (deep) inside region 1, its exit (deep) within region 2, see figure 2, and it operates in the following way: any

object entering a particular region, region 1, at time t, reappears in another region, 2, with a probability α but at time $t-T$, i.e. it has moved backwards in time. Similarly, an object entering region 2 at a time t will reappear in region 1 at time $t+T$ with a probability β, i.e. it has moved forward in time. This is the essence of what a time machine does, and is the only effect we are going to study in this paper. These two regions, 1 and 2, could contain the mouths of a wormhole, and we will often refer to them as the "mouths" of the time machine. No assumption is made concerning the actual structure of the time machine, it could be a wormhole or it could be something else.
The objects will be taken to be the quanta of some scalar field (one could with very little extra trouble – or gain – treat quanta of arbitrary spin too). It will be shown that the number of particles entering the wormhole is different from the number coming out in the other end which is most unfortunate. Thus time machines make it possible to distinguish past and future, by for instance looking at the density of some Bose field initially distributed homogeneously in space. They also pose a threat to energy conservation. Of course one could put this difference in particles/energy into the time machine's internal structure in order to have energy conservation - getting the extra particles out/in would thus be classifiable as part of the maintenance costs, but to an external observer a neglected time machine looks like an energy source/drain. If they are homogeneously distributed, this observation makes the existence of wormholes (or any other structure capable of supporting a time machine) with sizes in the interval between $\sim 10^{-18}$ m and $\sim 10^{8}$ m highly unlikely — they would have been observed. It also makes it dubious whether a "time machine" would really be up to its name, i.e. whether a space-time possessing closed time-like curves, would function as a time machine in the traditional sense of the word.

Breakdown of Unitarity in the Presence of Time Machines

We consider a partition of space, and we label each of these regions such that region 1 is one of the "mouths" and region 2 the other. We assume that particles entering region 1 will reappear, with some probability, in region 2 but at an earlier time and vice versa. Since the "mouths" are assumed to lie

deep within the appropriate regions, these probabilities, α, β will typically not be one, i.e. $\alpha, \beta < 1$. The time step will be assumed identical in both directions and will be denoted by T. This is not a severe assumption: if the time steps were different in the two directions, non-unitarity would be obvious. The Hamiltonian will be taken to be the simplest possible, namely a slight generalisation of the canonical Hamiltonian of a free field in number representation:

$$H = \alpha a_1^\dagger(t+T)a_2(t) + \beta a_2^\dagger(t-T)a_1(t) + g\sum_{i=1}^{N} a_i^\dagger(t)a_i(t) \tag{1}$$

with i labelling the various regions, $i = 1, 2, ..., N$ where N could be infinite (it has to be at least three: the two "mouths" and the rest of the universe). Here the g-term simply counts the number of quanta in the various regions at time t, whereas the α, β-terms describe the actual time machine effect.
Had i been the momentum and had t instead referred to a particular site in a chain, then this would be a familiar Hamiltonian – the first two terms would be "hopping terms" describing the possibility of a quanta to jump from one site to another.
We consider the regions 1 and 2 as identified modulo a time-shift, which implies the following commutator relations (assuming bosonic statistics)

$$\left[a_i(t), a_j^\dagger(t')\right] = \delta_{ij}\Delta(t-t') + \delta_{i1}\delta_{j2}\Delta(t'-t+T) + \delta_{i2}\delta_{j1}\Delta(t'-t-T) \qquad i, j = 1, 2, ..., N \tag{2}$$

the remaining commutators all vanishing.[1] The function Δ is a (possibly) smeared Dirac delta-distribution, the smearing mimicking some uncertainty in the values of t, t'. Its precise form matters little for our calculation; it could just as well be a proper Dirac delta-distribution. Our lack of knowledge about the precise structure of the time-machine can be parametrised by this function $\Delta(t)$ and the coefficients α, β appearing in the Hamiltonian. So the second quantisation operators corresponding to different regions at different

[1] Thus the time machine gives rise to two modifications, (1) the presence of the α, β-terms in the Hamiltonian, and (2) the $\Delta(t' - t \pm T)$-terms in the commutator relations. These two modifications are of course not independent: putting either α or β equal to zero amounts to forbidding travel through the wormhole in the corresponding direction, and hence the analogous term in the commutator relations should also be removed. To avoid a too heavy notation, we have decided, however, not to let this appear explicitly in equations (1) and (2).

time commute, except for those corresponding to the “mouths”.
The time evolution operator $U(t,t')$ is given by $U(t,t') = U(t-t') = e^{-iH(t-t')}$ and hence we need to evaluate powers of H. We want to find the matrix elements of $U(t,t')$. Denoting the states by $|n,t\rangle$, with $n = (n_1, n_2, ..., n_N)$ a multi index describing the number of quanta in each region, we have

$$\begin{aligned}
\langle n,t|H|n',t'\rangle &= \alpha\delta_{n_2',n_2-1}\delta_{n_1',n_1+1}\Delta(t'-t-T)\sqrt{n_2(n_1+1)}\prod_{i\neq 1,2}\delta_{n_i',n_i} + \\
&\quad \beta\delta_{n_2',n_2+1}\delta_{n_1',n_1-1}\Delta(t'-t+T)\sqrt{n_1n_2}\prod_{i\neq 1,2}\delta_{n_i',n_i} + \\
&\quad g\Delta(t-t')\delta(n,n')\sum_i n_i \qquad (3)
\end{aligned}$$

where $\delta(n,n') \equiv \prod_i \delta_{n_i,n_i'}$ is a Kronecker delta.
Similarly we get

$$\begin{aligned}
\langle n,t|H^2|n',t'\rangle &= \alpha^2\delta_{n_1',n_1-2}\delta_{n_2',n_2+2}\sqrt{(n_1-2)(n_1-3)(n_2+1)(n_2+2)}\Delta(t'-t+T)\delta_{12}(n,n') \\
&\quad +\alpha^2\delta_{n_1',n_1-1}\delta_{n_2',n_2+1}\sqrt{(n_1-1)(n_2+1)}\Delta(t'-t+T)\delta_{12}(n,n') \\
&\quad +\beta^2\delta_{n_1',n_1+2}\delta_{n_2',n_2-2}\sqrt{(n_1+3)(n_1+4)n_2(n_2-1)}\Delta(t'-t-T)\delta_{12}(n,n') \\
&\quad +\beta^2\delta_{n_1',n_1+1}\delta_{n_2',n_2-1}\sqrt{(n_1-1)(n_2+1)}\Delta(t'-t-T)\delta_{12}(n,n') \\
&\quad +g^2\delta(n,n')\Delta(t-t')\sum_{i\neq j} n_in_j + g^2\delta(n,n')\Delta(t-t')\sum_i n_i(n_i+1) \\
&\quad +\alpha\beta\left(n_2(n_1+1)+n_1n_2\right)\delta_{12}(n,n')\delta(|t-t'|-T) \\
&\quad +\alpha g\delta_{12}(n,n')\sum_i n_i\left(\delta_{n_1',n_1-1}\delta_{n_2',n_2+1}\Delta(t'-t+T)\sqrt{(n_1-1)(n_2+1)}+1\right) \\
&\quad +\beta g\sum_i n_i\left(\delta_{n_1',n_1+1}\delta_{n_2',n_2-1}\Delta(t'-t-T)\sqrt{(n_2-1)(n_1+1)}+1\right) \qquad (4)
\end{aligned}$$

with$\delta_{12}(n,n') \equiv \prod_{i\neq 1,2}\delta_{n_i',n_i}$. The time asymmetry of the Hamiltonian thus manifests itself in the evolution operator. This will be seen even more clearly in the next order contribution.
A convenient way of representing the various contributions are in terms of diagrams as follows: the two regions 1 and 2 are represented by two dots, • – the remaining $N-2$ regions need not be drawn, as they are not influenced by the time machine – the particle motion is then indicated by arrows, the g-terms counts the number of particles and are essentially vacuum terms,

they are represented by closed loops. This gives the diagrams listed in table 1. We refer to these as "worm tracks" (again thinking of the time machine as being made from a wormhole). Table 2 shows the various contributions to H^3 (here $t_\pm \equiv t \pm T$). We see that we generate asymmetries even in these *low order* terms. The worm tracks and the weights with which they appear are listed in table 3.
The Hamiltonian itself, is of course not a symmetric operator, as it identifies two different regions provided there is specific difference between the times, but when calculating the higher powers of H we discover new asymmetries, which were not to be expected *a priori*. This is so even in the most symmetric case $\beta = \alpha$, in fact the result is quite independent of what the precise values of the parameters α, β, g are.
It follows from eqs(3,4) and tables 2-3 that more quanta are exiting the time machine than there are entering it. The non-symmetric nature of the Hamiltonian thus generates, through the time evolution operator, an irreversibility, which is surprisingly strong.

Generation of Entropy

Non-symmetric time evolution is usually taken to be a sign of irreversibility and hence of entropy generation. We want to show that this is certainly so in our case, at least to the very lowest order.
Given a density matrix, ρ, the entropy is

$$S = -\mathrm{Tr}\ \rho \ln \rho \tag{5}$$

In our case ρ is (up to a normalisation constant) just the time evolution operator $U(t, t')$. Thus we can use our expressions for the matrix elements of the Hamiltonian found above. First of all we notice that the terms only involving the g-contributions correspond to a free field configuration and consequently have vanishing entropy change (if all the contributions are added together). We only need to concentrate on the contributions involving the α, β-part. This is also what one would expect, as these are precisely the time machine specific parts of the Hamiltonian. Furthermore, since a trace is involved in the definition of S we only need to keep the diagonal parts of $\langle n, t|H^k|n', t'\rangle$. Since $\rho \ln \rho \sim UH$, the first such contribution comes from the

matrix elements of H^2. Thus

$$\begin{aligned} \mathrm{Tr} UH &= g\Delta(t-t')\sum_n n - (t-t')\left(g^2\Delta(t-t')\sum_{n_i,n_j} n_i n_j + \right. \\ & \left. \alpha\beta\Delta(|t-t'|-T)\sum_{n_1,n_2}(n_2(n_1+1)+n_1n_2)+\ldots\right) \end{aligned} \quad (6)$$

The only surviving term is seen to be (as mentioned above, the g-terms will vanish when one takes all powers into account)

$$(t-t')\alpha\beta\Delta(|t-t'|-T)\sum_{n_1,n_2}(n_2(n_1+1)+n_1n_2) \quad (7)$$

Now, this sum is divergent and need to be regularized. The obvious regularization scheme to choose is ζ-function regularization, (Hawking (1977), Ramond (1989)). One replaces sums like

$$\sum_n n^{-s}$$

by a Riemann ζ-function, $\zeta(s)$. This can be analytically continued to values of s where the above, unregularized summation is illdefined.
In our case we need

$$\left(\sum_n n\right)_{\mathrm{reg}} = \zeta(-1) = -\frac{1}{12} \quad (8)$$

and similarly

$$\left(\sum_n (n+1)^{-s}\right)_{\mathrm{reg}} = \zeta(s,1) \quad (9)$$

where $\zeta(s,a)$ is the so-called Hurwitz ζ-function. We only need to know the value at $a=1, s=-1$, corresponding to a regularized value for $\sum_{n_1}(n_1+1) := \zeta(-1,1) = -\frac{1}{12}$. Hence the regularized contribution to the entropy reads

$$(t-t')\alpha\beta\Delta(|t-t'|-T)\sum_{n_1,n_2}(n_2(n_1+1)+n_1n_2) := \frac{1}{72}(t-t')\alpha\beta\Delta(|t-t'|-T) \quad (10)$$

Whenever $\alpha\beta > 0$ this is positive, and hence we have created entropy.
Hence, time-machines can generate entropy and will consequently generate an arrow of time, contrary to what one would expect.

Time-Evolution of Operators and Generalised Bogulyubov Transformations

From the commutator relations it is straightforward to derive the equations of motion for the operators $a_1, a_1^\dagger, a_2, a_2^\dagger$. These turn out to be

$$\begin{aligned} i\dot{a}_1(t) &= -(\beta+g)a_1(t) && (11)\\ i\dot{a}_1^\dagger(t) &= \beta a_2^\dagger(t-T) + g a_1^\dagger(t) && (12)\\ i\dot{a}_2(t) &= -(\alpha+g)a_2(t) && (13)\\ i\dot{a}_2^\dagger &= \alpha a_1^\dagger(t+T) + g a_2^\dagger(t) && (14) \end{aligned}$$

in which the asymmetry is also apparent. We can diagonalise these by means of a generalised Bogulyubov transformation. Write

$$\begin{aligned} b_1^\dagger(t) &= U_{11}(t)a_1^\dagger(t) + U_{12}(t)a_2^\dagger(t-T) && (15)\\ b_2^\dagger(t) &= U_{21}(t)a_1^\dagger(t+T) + U_{22}(t)a_2^\dagger(t) && (16) \end{aligned}$$

while the annihilation operators are not transformed. The transformation matrix $U(t)$ then has to satisfy

$$i\frac{d}{dt}\begin{pmatrix} U_{11} \\ U_{12} \end{pmatrix} = \begin{pmatrix} \omega_1 - g & -\alpha \\ -\beta & \omega_1 - g \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{12} \end{pmatrix} \qquad (17)$$

$$i\frac{d}{dt}\begin{pmatrix} U_{21} \\ U_{22} \end{pmatrix} = \begin{pmatrix} \omega_2 - g & -\alpha \\ -\beta & \omega_2 - g \end{pmatrix}\begin{pmatrix} U_{21} \\ U_{22} \end{pmatrix} \qquad (18)$$

where ω_1, ω_2 are the energies. Solving these equations is an easy matter (the coefficients $\omega_1, \omega_2, \alpha, \beta, g$ are all constants). The new operators then satisfy $\dot{b}_i^\dagger = -i\omega_i b_i^\dagger,\ i = 1, 2.$

Thus, considering the four operators $a_i, a_i^\dagger$ as independent, we can make a transformation, unto "normal modes", $b_i^\dagger, a_i$, the energies of which are $\omega_1, \omega_2, -(\alpha+g), -(\beta+g)$. This means that we *can* transform the Hamiltonian unto a diagonal form, using a kind of generalised normal modes, but these modes will manifestly break hermiticity, as then $(a_i)^\dagger = a_i^\dagger \neq b_i^\dagger$ – the quanta annihilated by a_i are not the same as those created by $b_i^\dagger$. This is also seen in the fact that the "energies" of the operators $b_i^\dagger$ (i.e. ω_1, ω_2) need not be identical to that of the a_i (i.e. $-(\alpha+g), -(\beta+g)$). Since "switching off" the time machine forces $(a_i)^\dagger = b_i^\dagger$ and the energies to be identical, the

time-machine is then seen as a mechanism that forces $b_i^\dagger$ away from $(a_i)^\dagger$ for $i = 1, 2$ (or equivalently as driving ω_i away from $-(\alpha+g), -(\beta+g))$, thereby generating non-unitarity of the time-evolution operator. We note that in this "diagonalised" representation of the Hamiltonian, the explicit reference to the time-shift T has disappeared; it will only enter if one transforms back to the original basis.

Conclusion

We assumed the existence of some kind of cosmic time (the 3+1 splitting) at least sufficiently far away from regions 1 and 2. But this cosmic time will *a priori* not have a particular direction – both the laws of relativity and of quantum mechanics are invariant under time-reflections. It is therefore rather surprising that the presence of time machines, that above all is seen as destroying causality, creates an irreversibility and thus, to be consistent with the second law of thermodynamics, imposes the arrow of time.

However, this is not the only physical effect of such time machines. Also basic subjects of physics are influenced on top of the problems with causality. Notably, in quantum field theory unitarity is broken (this is actually due to the breakdown of causality) and renormalisation theory will need a modification due to the emergence of topologically in-equivalent loop-diagrams, some of which it is not *a priori* possible to do away with as they stem from the breakdown of causality.

There is also a problem with the conservation of energy. Since more quanta are leaving than entering the time machine regions, energy has to supplied in order to have energy conservation. This need for constantly supplying energy will, quite irrespective of the problems of actually avoiding the energy from traversing the time machine, thus exacerbates the maintenance cost making them even more unstable than previously thought (Antonsen, Bormann (1995 and 1996)).

We emphasise that these conclusions are quite generic as any time machine will, from a bird's eye view, behave as the model presented here.

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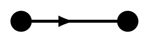
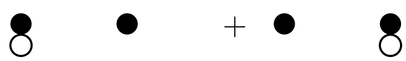
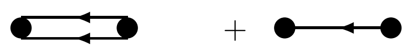
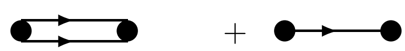
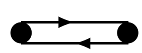
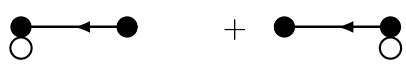
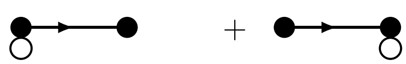
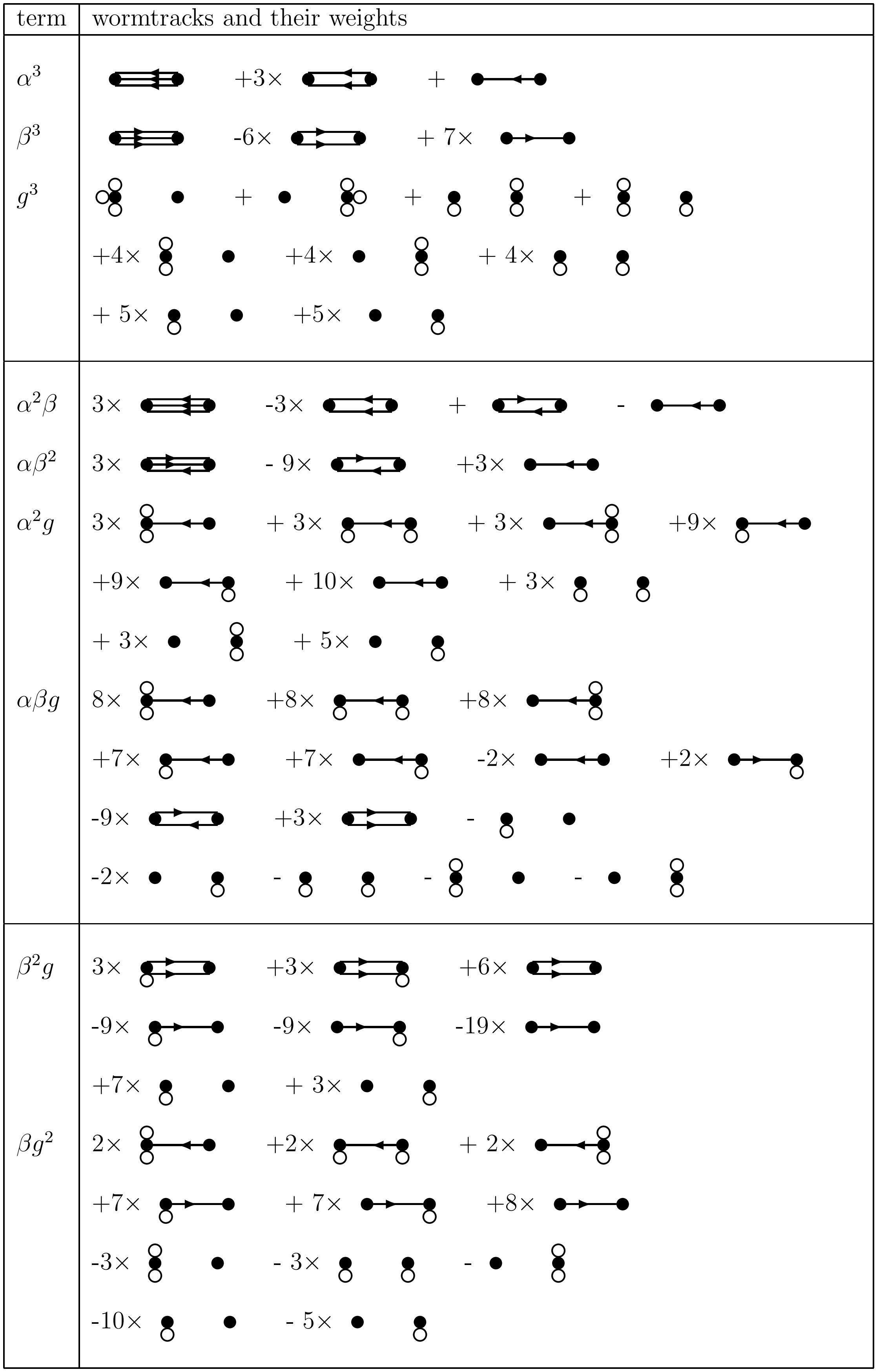
power of H	term	wormtracks
	α	
H	β	
	g	
	α^2	
	β^2	
	g^2	
H^2	$\alpha\beta$	
	αg	
	βg	

Table 1: The wormtracks corresponding to the various contributions to H and H^2. The filled out circles represents the regions 1 and 2 respectively, while open circles represent a number operator and arrows a motion of a particle as described in the text.

term	contribution to H^3
α^3	$a_1^\dagger(t_+)a_1^\dagger(t_+)a_1^\dagger(t_+)a_2a_2a_2 + 3a_1^\dagger(t_+)a_1^\dagger(t_+)a_2a_2 + a_1^\dagger(t_+)a_2$
β^3	$a_1a_1a_1a_2^\dagger(t_-)a_2^\dagger(t_-)a_2^\dagger(t_-) - 6a_1a_1a_2^\dagger(t_-)a_2^\dagger(t_-) + 7a_1a_2^\dagger(t_-)$
g^3	$\sum_{ijk} a_i^\dagger a_j^\dagger a_k^\dagger a_ia_ja_k + 3\sum_{i,j} a_i^\dagger a_j^\dagger a_ia_j + \sum_i n_i$
$\alpha^2\beta$	$3a_1^\dagger(t_+)a_1^\dagger(t_+)a_2a_2a_1a_2^\dagger(t_-) - 3a_1^\dagger(t_+)a_1^\dagger(t_+)a_2a_2 + a_1^\dagger(t_+)a_2a_1a_2^\dagger(t_-) - a_1^\dagger(t_+)a_2$
$\alpha\beta^2$	$3a_1^\dagger(t_+)a_2a_1a_1a_2^\dagger(t_-)a_2^\dagger(t_-) - 9a_1^\dagger(t_+)a_2a_1a_2^\dagger(t_-) + 3a_1^\dagger(t_+)a_2$
$\alpha^2 g$	$3\sum_j a_1^\dagger(t_+)a_1^\dagger(t_+)a_j^\dagger a_2a_2a_j + 3a_1^\dagger(t_+)a_1^\dagger(t_+)a_2a_2 + 2\sum_j a_1^\dagger(t_+)a_j^\dagger a_ja_2 +$
	$3a_1^\dagger(t_+)a_2^\dagger a_2a_2 + a_1^\dagger(t_+)a_2 + n_2$
αg^2	$3\sum_{jk} a_1^\dagger(t_+)a_j^\dagger a_k^\dagger a_2a_k + 6\sum_j a_1^\dagger(t_+)a_j^\dagger a_2a_j + a_1^\dagger(t_+)a_2 + 3\sum_j a_2^\dagger a_j^\dagger a_ja_2 + 2n_2$
$\alpha\beta g$	$8\sum_j a_1^\dagger(t_+)a_j^\dagger a_2a_1a_ja_2^\dagger(t_-) - 9a_1^\dagger(t_+)a_2 - 2a_2^\dagger a_1 - n_2 - 9a_1^\dagger(t_+)a_2a_1a_2^\dagger(t_-) -$
	$3a_1^\dagger(t_+)a_1^\dagger a_2a_2 + a_2^\dagger a_2a_2a_2^\dagger(t_-) - \sum_j a_1^\dagger(t_+)a_j^\dagger a_ja_1 + 2a_2^\dagger a_2a_1a_2^\dagger(t_-) - \sum_j a_1^\dagger a_j^\dagger a_ja_2$
$\beta^2 g$	$3\sum_j a_j^\dagger a_1a_ja_1a_2^\dagger(t_-)a_2^\dagger(t_-) - 2a_1^\dagger a_1a_2a_2^\dagger(t_-) - 9\sum_j a_j^\dagger a_ia_ja_2^\dagger(t_-) +$
	$3a_aa_1a_2^\dagger(t_-)a_2^\dagger(t_-) - 10a_1a_2^\dagger(t_-) + 4n_1 + 3\sum_j n_j + 3$
βg^2	$2\sum_{jk} a_j^\dagger a_k^\dagger a_1a_ja_ka_2^\dagger(t_-) - 2\sum_j a_j^\dagger a_1^\dagger a_1a_j + 5\sum_j a_j^\dagger a_1a_ja_2^\dagger(t_-) -$
	$\sum_{jk} a_j^\dagger a_k^\dagger a_ja_k - 4\sum_j n_j - 3n_1 + a_1a_2^\dagger(t_-) - 1$

Table 2: The contributions to H^3. Only shifted times, $t_\pm = t \pm T$, are written explicitly. Some of these terms will have vanishing matrix elements.


term
wormtracks and their weights
α³
+3×
+
β³
-6×
+ 7×
g³
+
+
+
+4×
+4×
+ 4×
+ 5×
+5×
α²β
3×
-3×
+
-
αβ²
3×
- 9×
+3×
α²g
3×
+ 3×
+ 3×
+9×
+9×
+ 10×
+ 3×
+ 3×
+ 5×
αβg
8×
+8×
+8×
+7×
+7×
-2×
+2×
-9×
+3×
-
-2×
-
-
-
β²g
3×
+3×
+6×
-9×
-9×
-19×
+7×
+ 3×
βg²
2×
+2×
+ 2×
+7×
+ 7×
+8×
-3×
- 3×
-
-10×
- 5×